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Note

# On the Erdős-diameter of sets

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**Abstract**

Let $\delta(n)$ denote the minimum diameter of a set of $n$ points in the plane in which any two positive distances, if they are different, differ by at least one. Erdős conjectured that for $n$ sufficiently big we have $\delta(n) = n - 1$, the extremal configuration being $n$ equidistant points on a line. In this note we prove an asymptotic version of this conjecture for the special case of sets which lie in a parallel half-strip.

## 1. Introduction

In 1981 Erdős [5] asked for the minimum diameter $\delta(n)$ of a set of $n$ points in the plane in which any two occurring positive distances, if they are different, differ by at least one. In the following we call this distance property the Erdős property, and for any finite set $S$ the diameter of the smallest rescaled copy $\lambda S$ with this property the Erdős diameter $\delta(S)$.

Erdős gave a lower bound $\delta(n) > cn^{2/3}$ which was subsequently improved by Kanold [10] to $\delta(n) > \frac{1}{4}n^{3/4}$. In [9] Kanold generalized his argument to higher dimensions and obtained a lower bound of $(1/\sqrt{\frac{3}{2}d})n^{1/d}$ in the $d$-dimensional case, $(1/\sqrt{14})n^{1/2}$ in $\mathbb{R}^3$ and $0.366n^{3/4}$ in the plane. Chung et al. [3] proved in 1992 that any set of $n$ points in the plane contains at least $c_1(\log n)^{-c_2}n^{4/5}$ different distances; this gives the currently best lower bound

$$\delta(n) > c_1(\log n)^{-c_2}n^{4/5}.$$

The best general upper bound in the plane is given by $n$ equidistant points on a line; this led Erdős to the conjecture that $\delta(n) = n - 1$ for $n$ sufficiently big [6–8]. In $\mathbb{R}^d$ ($d \geqslant 2$) suitable lattice subsets give an upper bound of order $n^{2/d}$.

Piepmeyer [11] obtained in 1992 as a byproduct of his complete classification of planar sets with up to three different distances the exact values

$$\delta(4) = 2, \qquad \delta(5) = \tfrac{1}{2}(\sqrt{5} + 3), \qquad \delta(6) = 2 + \sqrt{2};$$

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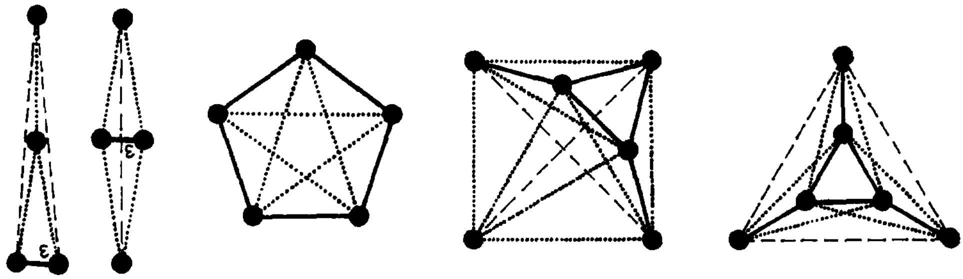

Fig. 1.

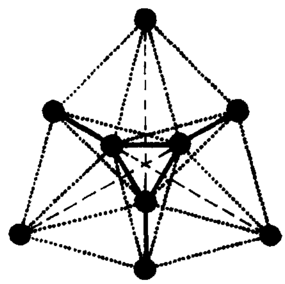

Fig. 2.

all external configurations are given in Fig. 1 (for four points we must take the infimum for $\varepsilon \to 0$). He noted that the nine-point four-distance configuration of Fig. 2 gives an upper bound

$$\delta(7) \leqslant \delta(8) \leqslant \delta(9) \leqslant (1 + \sqrt{2})\sqrt{2 + \sqrt{3}} = 4.6639\ldots$$

This configuration is also interesting in several other respects: it is contained as a building-block in all sufficiently large sets with maximum number of second-smallest distances [2] and it occurs in Schade's list [13] as the unique type of nine-point configuration with the maximum possible number 18 of unit distances (taking the second-smallest distance as unit distance), furthermore there is no set of nine points with exactly 17 unit distances.

In the statement of his conjecture Erdős [8] made the additional assumption that the smallest occurring distance should be at least one. Piepmeyer [12] shows this assumption to be unnecessary. Using that in a set of $n$ points with the Erdős property which contains two points with distance smaller that one all remaining points must be equidistant to these two points, he shows that for $n \geqslant 5$ such a set has a diameter greater than $n - 1$.

Baron [1] considers the case of the smallest distance being exactly one, but in arbitrary dimension. With this additional assumption he shows that in the plane $n$ equidistant points on a line are the unique minimum-diameter configuration.

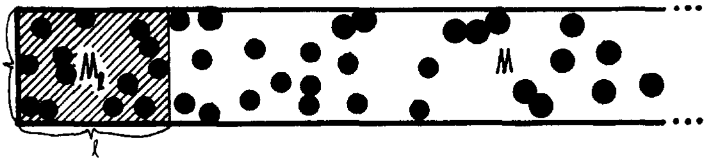


Fig. 3.

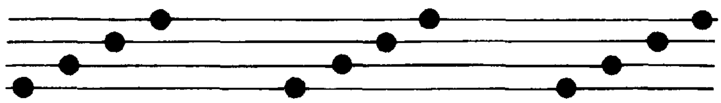

Fig. 4.

## 2. The result

We study the following situation: Let $M$ denote an infinite discrete subset of the half-strip $[0, \infty[ \times [0, h]$ and $M_l$ the subset contained in $[0, l] \times [0, h]$ $(h > 0, l \geqslant 0)$. Moreover, we assume that there is a constant $\alpha > 0$ such that $|M_l| \leqslant \alpha l$ for $l$ sufficiently big (Fig. 3). We prove the following theorem.

**Theorem.** $\limsup_{l \to \infty} [\delta(M_l)/|M_l|] \geqslant 1$, *where equality holds if and only if all points lie on a horizontal line and their first coordinates form a density-*1 *subset of an arithmetic sequence.*

In case of equality we have $\delta(M_l) \geqslant |M_l| - 1$ for $|M_l| \geqslant 3$. The theorem is best possible in the sense that for any $\varepsilon > 0$ we have another example with $\limsup_{l \to \infty} [\delta(M_l)/|M_l|] < 1 + \varepsilon$, consisting of equidistant points alternating on two parallel lines with small distance. We believe this to be a special case of the following conjecture.

**Conjecture.** If $\limsup_{l \to \infty} [\delta(M_l)/|M_l|] \leqslant 2k/(k + 1)$ then $M$ can be covered by $k$ parallel lines. Extremal configurations would be sets like Fig. 4.

## 3. Proof

Let $l_i$ denote the smallest $l$ such that $M_l$ contains at least $i$ points and $\beta_i$ the smallest positive difference of positive distances occurring in $M_{l_i}$ ($\beta_i$ exists at least for $i \geqslant 4$). Then we have $\delta(M_{l_i}) \geqslant \beta_i^{-1}(l_i - l_1)$ for all $i \geqslant 4$, so by $|M_l| \leqslant \alpha l$ and supposing $\limsup_{l \to \infty} [\delta(M_l)/|M_l|] < \infty$ we obtain $\beta := \inf_i \beta_i > 0$.

**Lemma.** *If* $\limsup_{l \to \infty} [\delta(M_l)/|M_l|] < \infty$ *then all but finitely many points of* $M$ *have distinct first coordinates, and all the distinct coordinates differ by at least* $\beta$. *If*

$\limsup_{l\to\infty}[\delta(M_l)/|M_l|] < 2$ *then also the different positive differences of first coordinates differ by at least* $\beta$.

**Proof of the Lemma.** Let $A, B \in M$ be two points with first coordinates $x_A, x_B$ and $(P_i)_{i=1}^{\infty}$ be the sequence of points in $M$ so that $P_i$ has first coordinate $l_i$. Since $\lim_{i\to\infty}|d(P_i, A) - d(P_i, B)| = |x_A - x_B|$ we have either $|x_A - x_B| \geqslant \beta$ or $x_A = x_B$ and $d(P_i, A) = d(P_i, B)$ for $i$ sufficiently big. So if $x_A = x_B$ all but a finite number of points of $M$ lie on the middle perpendicular of $\overline{AB}$ and have therefore distinct first coordinates.

To prove the second statement let $A_1, B_1, A_2, B_2 \in M$ be four points with first coordinates $x_{A_1}, x_{B_1}, x_{A_2}, x_{B_2}, x_{B_k} > x_{A_k}$, such that $|(x_{B_1} - x_{A_1}) - (x_{B_2} - x_{A_2})| = \beta - \varepsilon$ for some $\varepsilon > 0$. Let $(d_{i,k})_{i=1}^{\infty}$ for $k = 1, 2$ denote the sequences of distances $d(P_i, B_k)$. For each $k$ and sufficiently big $i$ these distances are distinct and form a subsequence of all occurring distances. If $(d_{i=1})_{i=1}^{\infty}$ and $(d_{i,2})_{i=1}^{\infty}$ had only finitely many elements in common then for sufficiently big $i$ each $M_{l_i}$ would contain at least $2i - c$ distinct positive distances, so that $\limsup_{l\to\infty}[\delta(M_l)/|M_l|] \geqslant 2$. Thus there is an infinite sequence $(d_m^*)_{m=1}^{\infty}$ such that for each $m$ there exist $i_1(m)$, $i_2(m)$ with $d_m^* = d(P_{i_1(m)}, B_1) = d(P_{i_2(m)}, B_2)$. For sufficiently big $m$ and $k = 1, 2$ we have $|(d(P_{i_k(m)}, A_k) - d(P_{i_k(m)}, B_k)) - (x_{B_k} - x_{A_k})| < \varepsilon/2$, so $|d(P_{i_1(m)}, A_1) - d(P_{i_2(m)}, A_2)| < \beta$. Therefore for $m$ sufficiently big we have $d(P_{i_1(m)}, A_1) = d(P_{i_2(m)}, A_2)$ and

$$x_{B_1} - x_{A_1} = \lim_{m\to\infty} d(P_{i_1(m)}, A_1) - d(P_{i_1(m)}, B_1) = \lim_{m\to\infty} d(P_{i_2(m)}, A_2) - d(P_{i_2(m)}, B_2)$$

$$= x_{B_2} - x_{A_2},$$

and the Lemma is proved. □

**Proof of the Theorem.** Let $\gamma$ denote the infimum of positive differences of first coordinates of points in $M$. By the first part of the Lemma we have $\gamma \geqslant \beta$. Since $\delta(M_{l_i}) \geqslant \beta_i^{-1}\gamma(i - c)$ for some $c \geqslant 1$ and all $i$, we have $\limsup_{i\to\infty}[\delta(M_{l_i})/i] \geqslant \gamma/\beta \geqslant 1$.

For $\limsup_{l\to\infty}[\delta(M_l)/|M_l|] = 1$ it is therefore necessary that $\gamma = \beta$. By the second part of the Lemma the set of differences of first coordinates is discrete, so there are consecutive points whose difference of first coordinates is $\beta$. Whenever the difference of first coordinates of two consecutive points is larger than $\beta$, it is at least $2\beta$. Since $\limsup_{l\to\infty}[\delta(M_l)/|M_l|] = 1$ the set of larger intervals has upper density 0, so for each $k$ there are $k + 1$ consecutive points with differences of first coordinates $\beta$. Therefore for each $k$ there occurs in $M$ a distance $d_k = k\beta + \varepsilon_k$ with $\varepsilon_k \geqslant 0$ and $\lim_{k\to\infty}\varepsilon_k = 0$. For some $k_0$, for each $k \geqslant k_0$ any such distance satisfies $k\beta \leqslant d_k < (k + 1)\beta$. Since any two different positive distances in $M$ differ by at least $\beta$, for $k \geqslant k_0$ the sequence $(\varepsilon_k)_{k=k_0}^{\infty}$ is also an increasing sequence. Therefore, $\varepsilon_k = 0$ holds for all $k \geqslant k_0$. Hence, for any $k \geqslant 2k_0$ any $k + 1$ consecutive points with differences of consecutive first coordinates $\beta$ have equal second coordinates. Therefore, the set of positive distances equals $\{\beta, 2\beta, 3\beta, \ldots\}$, and hence for any $A, B \in M$ with

$x_B - x_A = \beta$, $y_B = y_A$, any sufficiently far point $C \in M$ is collinear with $A$, $B$, and $x_C - x_B$ is an integer multiple of $\beta$. Choosing now another pair $A', B' \in M$ with the same properties, but with $x_{A'}$ sufficiently large, we get the same conclusion for any point $C \in M$ for which $x_C$ is below an arbitrary fixed bound. This readily implies the statement of the Theorem. $\square$

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